Preface

This is a compendium of problems from the "Master's Review Examinations" for physics graduate students at the University of Washington. This compendium covers the period after Autumn 2011 when the Department changed the format from a classic stand alone Qualifying Exam, (held late Summer and early Spring) into the current course integrated 'Masters Review Exam (MRE) format. The problems from the pre Autumn 2011 period can be found in the separate Physics Qualfying Exam Problems compendium.

UW physics Ph.D. students are strongly encouraged to study *all* the problems in these two compendia. Students should not be surprised to see a mix of new and old problems on future exams. The level of difficulty of the problems on the old Qualifying Exams and the new Masters Review Exams is the same.

Problems are grouped by year, and ordered as: Thermodynamics and Statistical Mechanics, Classical Mechanics, Quantum Mechanics, and Electromagnetism.

Here are bits of advice that were given to the students studying for the stand alone old version of qualifying exam, most of it still applies for the current MRE format:

- Try to view your time spent studying for the Qual as an opportunity to integrate all the physics you have learned (and not just as a painful externally imposed burden).
- Read problems in their entirety first, and try to predict qualitatively how things will work out before doing any calculations in detail. Use this as a means to improve your physical intuition and understanding.
- Some problems are easy. Some are harder. Try to identify the *easiest* way to do a problem, and don't work harder than you have to. Make yourself do the easy problems *fast*, so that you will have more time to devote to harder problems. Make sure you *recognize* when a problem is easy.
- Always include enough explanation so that a reader can understand your reasoning.
- At the end of every problem, or part of a problem, look at your result and ask yourself if there is any way to show quickly that it is *wrong*. Dimensional analysis, and consideration of simplifying limits with known behavior, are both enormously useful techniques for identifying errors. Make the use of these techniques an ingrained habit.
- Recognize that good techniques for studying Qual problems, such as those just mentioned, are also good techniques for real research. That's the point of the Qual!

Notation

Boldface symbols like $\boldsymbol{r}$ or $\boldsymbol{k}$ denote three-dimensional spatial vectors. Unit vectors pointing along coordinate axes are denoted as $\hat{\boldsymbol{e}}_x$, $\hat{\boldsymbol{e}}_y$, *etc.* Carets are sometimes (but not always) placed over quantum operators to distinguish them from c-numbers. Dots are sometimes used as shorthand for time derivatives, so $\dot{f} \equiv df/dt$. Implied summation conventions are occasionally employed.

Physical constants appearing in various problems include:

c	vacuum speed of light
e	electron charge
m_e	electron mass
m_p	proton mass
ϵ_0	vacuum permittivity
μ_0	vacuum permeability
$Z_0 \equiv \mu_0 c$	vacuum impedance
$h \equiv 2\pi\hbar$	Planck's constant
g	Earth's gravitational acceleration
G_{N}	Newton gravitational constant
k_{B}	Boltzmann's constant

Trying to memorize SI values of all these constants is *not* recommended. It is much more helpful to remember useful combinations such as:

$\alpha \equiv e^2/(4\pi\epsilon_0\hbar c) \approx 1/137$	fine structure constant
$(300\,\mathrm{K})\,k_{\mathrm{B}} \approx \frac{1}{40}\,\mathrm{eV}$	room temperature
$m_e\,c^2 \approx 0.5\,\mathrm{MeV}$	electron rest energy
$m_p\,c^2 \approx 1\,\mathrm{GeV}$	proton rest energy
$a_{\mathrm{B}} \equiv \hbar/(\alpha m_e c) \approx 0.5\,\text{Å}$	Bohr radius
$\frac{1}{2}\alpha^2 m_e c^2 \approx 13.6\,\mathrm{eV}$	Rydberg energy
$\hbar\,c \approx 200\,\mathrm{MeV\ fm}$	conversion constant
$1/\epsilon_0 = \mu_0\,c^2 \approx 10^{11}\,\mathrm{N\,m^2/C^2}$	conversion constant
$m_{\mathrm{Pl}} \equiv \sqrt{\hbar c/G_{\mathrm{N}}} \approx 10^{19}\,\mathrm{GeV}/c^2 \approx 0.2\,\mu\mathrm{g}$	Planck mass

1 classical ideal gas in a central potential

[**50** points]

A cloud of dilute classical mono-atomic gas molecules is confined to a region of 3 dimensional space by a central potential of the form $V(\vec{r}) = Ar^3$, with r the distance from the force center at $\vec{r} = \vec{0}$, and with $A > 0$.

$$E = \sum_{i=1}^{N} \Big[\frac{\vec{p}_i^{\,2}}{2m} + V(\vec{r}_i)\Big]$$

A rigid spherical object of radius R_0 is located at the center, such that the gas cloud extends from $R_0 < r < \infty$.

A. [**10** points] Evaluate the canonical partition function.

B. [**10** points] Determine the heat capacity of the gas for when the radius R_0 and particle number N are held constant.

C. [**10** points] Determine the average volume $\langle V \rangle = \frac{4}{3}\pi \langle r^3 \rangle$ of the gas cloud.

D. [**10** points] Determine the pressure exerted by the gas onto the surface of the R_0-sphere.

E. [**10** points] Determine the chemical potential of the gas cloud.

2 one dimensional bose gas

[**50** points]

A non-relativistic dilute (ideal) gas of bosons moves freely along a wire of length L.

A. [**10** points] Derive that the single particle density of states of this one-dimensional gas is of the form $g(E) \sim VE^a$. Determine the exponent a.

B. [**15** points] Write down the grand-canonical partition function and show that the pressure p, the total number of particles N, and the total internal energy U, are of the form

$$p = (k_B T)^b \ F_1(\frac{\mu}{k_B T}) \ \ ; \ \ N/L = (k_B T)^c \ F_2(\frac{\mu}{k_B T}) \ \ ; \ \ U/L = (k_B T)^d \ F_3(\frac{\mu}{k_B T})$$

with F_1, F_2, and F_3 integrals that are functions of $\mu/k_B T$. Do not evaluate those integrals. Determine the exponents b, c, and d.

C. [**10** points] Derive that for this gas $pL = 2U$.

D. [**15** points] This gas undergoes a quasi-static adiabatic expansion to four times its original length L. Which fraction of its internal energy is expended as work?

1 circular motion in general central potential

[**30** points]

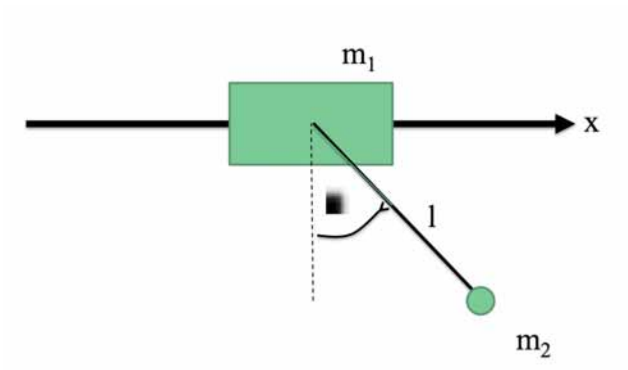
A particle of mass m moves in a central potential of the form $U(r) = -\frac{k}{r^n}$, where $k > 0$ and $n > 0$ are constants.

A. [**15** points] Find a radius of a circular orbit R as a function of its orbital momentum M, n, and k.

B. [**15** points] Find a criterium of stability of the circular orbit. (15 points.)

2 adiabatic theorem and pendulum

[**70** points]
Consider a simple pendulum of length l and mass m_2 which is attached to a mass m_1, which can move without friction horizontally along a bar. (See Fig.1.)



A. [**15** points] Find a Lagrangian of the system and the Lagrange equations.

B. [**15** points] How many vibrational modes does the system have? Find the eigenfrequency Ω of small amplitude oscillation.

C. [**20** points] Suppose the gravitational constant $g(t)$ changes in time slowly compared to the inverse eigenfrequency Ω^{-1}. In this case both the amplitude and the frequency of oscillations slowly change in time. Write the adiabatic invariant of the problem I in terms of the energy and the frequency the oscillations. Express the amplitude of the oscillations $\phi_0(t)$ in terms of I, l and $g(t)$. Assume that the amplitude of oscillations of the angle ϕ_0 is small.

D. [**20** points] Suppose that $g(t) = g_0 + g_1 \cos \gamma t$. Find an interval of frequencies γ where the parametric resonance takes place. Consider the case where $m_1 = \infty$ and the first mass does not move.

The following formulas may be useful:

$$\cos a \cos b = \frac{\cos(a-b) + \cos(a+b)}{2}$$
$$\sin a \cos b = \frac{\sin(a+b) + \sin(a-b)}{2}$$
$$\sin a \sin b = \frac{\cos(a-b) - \cos(a+b)}{2}$$

1 Two Spins in a Magnetic Field

Consider a system with 2 particles whose position is frozen. Particle A has spin 1/2, particle B has spin 1.

A. [**10** points] First consider the system in a strong, constant background magnetic field, so that the spin/spin interactions between the two particles can be neglected. The Hamiltonian for this system is given by

$$H = -\mu(\vec{S}_A + \vec{S}_B) \cdot \vec{B}$$

where μ is a constant. Find and list the energy eigenstates of this system.

B. [**25** points] The system is initially prepared in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\frac{1}{2}, 1\rangle + |-\frac{1}{2}, 1\rangle\right)$$

where the states $|m_A, m_B\rangle$ refer to eigenstates of S_{Az} and S_{Bz} (the z-components of the spin of A and B respectively) with eigenvalues m_A and m_B.

i. [**10** points] Simultaneous measurement of which two compatible observables could have produced this initial state? Explain.

ii. [**10** points] An experimentalist wants to apply a time-dependent external magnetic field (pointing in the z direction) in such a way that the state is changed into

$$|\tilde{\psi}\rangle = \frac{i}{\sqrt{2}}\left(-|\frac{1}{2}, 1\rangle + |-\frac{1}{2}, 1\rangle\right)$$

The magnetic field should vanish before time $t = 0$ and after a final time t_f. Also assume that the magnetic field is changed sufficiently slowly so that all induced electric fields can be neglected. What is the condition on the time dependent $\vec{B}$ to have this effect?

iii. [**5** points] Is the final state $|\tilde{\psi}\rangle$ an eigenstate of some components of $\vec{S}_A$ and $\vec{S}_B$? If so, which ones? Explain.

C. [**15** points] The two spins interact via a spin/spin interaction, giving rise to an additional term in the Hamiltonian:

$$\Delta H = -\gamma \vec{S}_A \cdot \vec{S}_B$$

where γ is a constant. Find the spectrum of eigenstates of the Hamiltonian, including the effects of the spin/spin interaction, in the presence of a constant background magnetic field .

2 Time Evolution

Consider a particle of mass m moving in a one-dimensional potential

$$V(x) = -\alpha\,\delta(x)\,,$$

where α is a positive constant.

A. [**15** points] Find the energy level(s) and the normalized wave function(s) of the bound state(s).

B. [**25** points] At time zero, the wavefunction of the particle (which is not necessarily an eigenfunction) is:

$$\psi(t{=}0, x) = A\,e^{-\beta|x|}\,.$$

with β being an arbitrary positive parameter not related to α.

- **i.** [**5** points] Explain qualitatively what happens to the wave function in the limit $t \to \infty$.
- **ii.** [**10** points] Find the probability $W(x)\,dx$ of finding the particle in the interval $(x, x + dx)$ in the limit $t \to \infty$.
- **iii.** [**5** points] Evaluate the integral $\int_{-L}^{L} dx\,W(x)$.
- **iv.** [**5** points] Consider the $L \to \infty$ limit of the integral you evaluated in part (iii). What is the physical interpretation of this quantity? Compare with the analogous quantity at $t = 0$ and qualitatively explain the result.

C. [**10** points] Now put the system in a box of width $2L$. That is the potential is as above for $|x| < L$, but $V = \infty$ for $|x| \geq L$. Qualitatively describe the spectrum of normalizable eigenstates in this case. How does this change affect the answer to problem B(iv)? Explain.

1 Electrostatic line charge with dielectrcum

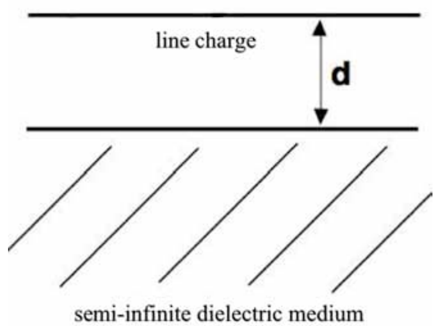
A. [**6** points] Write Maxwells equations for $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{H}$, and $\boldsymbol{B}$, in the presence of a free charge density ρ and free current density $\boldsymbol{j}$.

B. [**5** points] Define the electric displacement $\boldsymbol{D}$ in terms of the electric field $\boldsymbol{E}$ and polarization $\boldsymbol{P}$.

C. [**14** points] Consider an infinite line in empty space (in the absence of a dielectric) carrying a constant charge density λ located at $x = 0$, $y = 0$ and running in the z-direction. State Gauss's law and use it to find the electric field $\boldsymbol{E}$ a distance r from the line. Show that everywhere except at the line, the field $\boldsymbol{E}$ can be written in terms of a scalar potential, $\boldsymbol{E} = -\nabla\psi$, where (up to an additive constant)

$$\psi(\boldsymbol{x}) = -\frac{\lambda}{2\pi\epsilon_0}\,\mathrm{Re}\left[\log(x + iy)\right].$$

An infinitely long line of linear charge density λ is placed a distance d above a semi-infinite dielectric medium of permitivity ϵ, see figure.

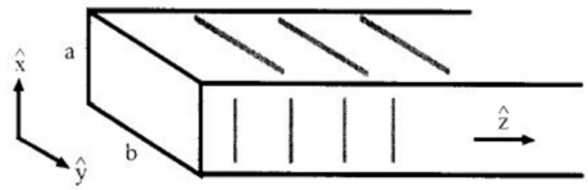


D. [**10** points] Given that there are no free charges or currents at the boundary of the dielectric medium, state the relations between components of $\boldsymbol{E}$ just above and just below the boundary. Then state these relations in terms of the scalar potential just above the boundary, $\psi_>$, and just below the boundary, $\psi_<$.

E. [**15** points] Determine the electric field $\boldsymbol{E}$ in the region above the dielectric.

2 Rectangular Waveguide TE Wave

A waveguide is formed from a rectangular cavity inside a **perfect** conductor. The cavity has sides of length a and b in the x and y directions with $b > a$ and has infinite extent in the z-direction. Further, the cavity is filled with a linear, homogeneous dielectric with permeability $\mu = \mu_0$ and permittivity ε.



Transverse electric (TE) traveling waves exist in the wave guide of the form

$$\boldsymbol{B}(x, y, z, t) = [B_{0x}(x, y)\,\hat{\boldsymbol{e}}_x + B_{0y}(x, y)\,\hat{\boldsymbol{e}}_y + B_{0z}(x, y)\,\hat{\boldsymbol{e}}_z]\; e^{i(kz-\omega t)}\,.$$

A. [**10** points] Specify the boundary conditions that $\boldsymbol{E}$ and $\boldsymbol{B}$ must satisfy at the interface between the conductor and dielectric.

B. [**10** points] Use Maxwell's equations to write B_{0x}, B_{0y}, E_{0x}, and E_{0y} in terms of B_{0z}.

C. [**10** points] Obtain the second-order partial differential equation that $B_{0z}(x, y)$ satisfies in the wave guide.

D. [**10** points] Solve the equation in part C to find solutions for $B_{0z}(x, y)$ in the wave guide which satisfy the boundary conditions of part A.

E. [**10** points] Find the lowest frequency of the TE_{mn} mode.

The following vector identities may be useful.

$$\begin{aligned}
\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) &= \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})\\
\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) &= (\boldsymbol{A}\cdot\boldsymbol{C})\,\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\,\boldsymbol{C}\\
(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D}) &= (\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D}) - (\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C})\\
\boldsymbol{\nabla}\times\boldsymbol{\nabla}\psi &= 0\\
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) &= 0
\end{aligned}$$

1 Fluctuating Pendulum (30 points total)

Consider a classical pendulum shown schematically in the figure below. Assume that it is in equilibrium with a thermostat.

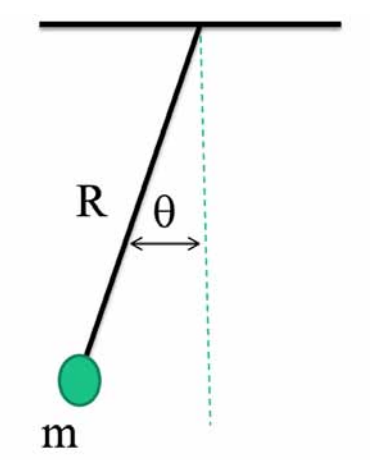
You may find useful the following Gaussian integrals:

$$\sqrt{\frac{\alpha}{(2\pi)}}\int_{-\infty}^{\infty}\exp(-\frac{1}{2}\alpha x^2)dx = 1 \quad \text{and} \quad \int_{-\infty}^{\infty}x^2\sqrt{\frac{\alpha}{(2\pi)}}\exp(-\frac{1}{2}\alpha x^2)dx = \frac{1}{\alpha}$$

A. [**10** points] Write an expression for the probability density to find the pendulum at the angle θ.

B. [**10** points] Calculate the variance of small fluctuations of the angle of the pendulum $\langle\theta^2\rangle$ in terms of R, m and the gravitational constant g.

C. [**10** points] State the equipartition theorem and calculate the value of the variance of the velocity $\langle v^2\rangle$. Here v is the velocity of the mass of the pendulum.



2 Two dimensional Electron Gas (70 points total)

Consider a non-interacting gas of electrons of mass m and energy $\epsilon_{\mathbf{p}} = \mathbf{p}^2/2m$ in 2 dimensions. The total number of electrons is N and the area of the sample is S.

A. [**10** points] Write the expression for the Fermi distribution of the occupation number of electrons $n_{\mathbf{p}}$ in terms of the chemical potential μ, temperature T, and the energy $\epsilon_{\mathbf{p}}$.

B. [**10** points] What is the sign of the chemical potential μ at $T = 0$? What is the sign of the chemical potential at high temperatures $T \gg E_F$ (Boltzmann gas)? Here E_F is the Fermi energy at $T = 0$.

C. [**10** points] Write an expression for the number of electrons N in terms of the Fermi momentum p_F.

D. [**10** points] Write an expression for the total electron energy E at $T = 0$ in terms of the number of electrons N and the area of the sample S.

E. [**10** points] Write an expression for the pressure P at $T = 0$ in terms of N and S.

F. [**10** points] Consider the case where there is a magnetic field $B_{\parallel}$ parallel to the 2D sample. Therefore it acts only on electron spins. Estimate the value of the magnetic field $B_{\parallel}^*$ which completely polarizes the gas at $T = 0$ in terms of the Fermi energy E_F and the Bohr magneton β. Here E_F is the Fermi energy in the absence of the magnetic field.

G. [**10** points] Write an expression for the Helmholtz free energy F at high temperatures $T \gg E_F$, where the translational degrees of freedom of electrons are classical (Boltzmann gas) and, using this result, calculate the heat capacity C_V. You do not have to calculate dimensionless integrals which do not contain physical parameters. (In this problem there is no external magnetic field.)

1 (40 points total)

A particle of mass m moves in a central potential of the form $U(r) = \quad \frac{\alpha}{nr^n}$, where $\quad > 0$ and $n < 2$. The Lagrangian written in the polar coordinates has a form

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\ \dot{}^2) + \frac{}{nr^n} \tag{1}$$

A. [**10** points] Write down expressions for the generalized momenta $p_\phi = M$, and p_r, for the Hamiltonian, for Hamiltonian equations, for the expression for the energy $E(M, r, \dot{r})$, and for the effective potential $U_{eff}(r)$.

B. [**10** points] Find the radius R of a circular orbit as a function of M, n and , and show that the orbit is stable.

C. [**10** points] Find a frequency ω of small oscillations of the radius of the orbit about the stable radius.

D. [**10** points] What is the condition for the oscillating trajectory to be closed?

2 (60 points total)

Consider a simple pendulum of length l and mass m whose point of support oscillates vertically according to the law $y = A \cos \Omega t$. (See Fig.1a.)

A. [**10** points] Derive the Lagrangian of the system. (It is convenient, but not necessary, to omit the total derivative.)

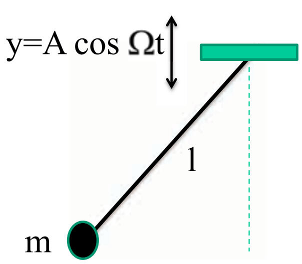
B. [**10** points] Write the Lagrange equation.

C. [**10** points] Suppose $\Omega \ll \sqrt{g/l}$ and the amplitude of the oscillation of the pendulum is small. So the equation of motion is linear. In this case both the amplitude and the frequency $\omega(t)$ of oscillation of the pendulum slowly change in time. Write the adiabatic invariant of the problem, I in terms of the energy E and ω. How does $E(t)$ depend on time?

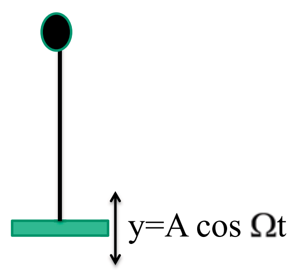
D. [**10** points] Find an interval of frequency $\Omega \pm \Delta\Omega$, where parametric resonance takes place. Do your calculations to lowest order in the amplitude A. Assume that the oscillations are linear.

E. [**10** points] Suppose now that $\Omega \gg \sqrt{g/l}$. Write the equation of motion and the effective potential averaged over the period of oscillation $2\pi/\Omega$.

F. [**10** points] What is the condition on the frequency for stability of the vertical inverted position of the pendulum, as shown in Fig.1b?



a)



b)

1 Symmetry in Quantum Mechanics (50 points total)

The parity operator in one dimension, P_1, reverses the sign of the position coordinate, x:

$$P_1\psi(x) = \psi(-x).$$

A. [**10** points] Find the eigenvalues and eigenfunctions of the parity operator. Is the parity operator Hermitian? Explain.

B. [**10** points] Calculate the commutator of P_1 with the position operators, $[P_1, \hat{x}]$, and with the position operator squared, $[P_1, \hat{x}^2]$.

C. [**10** points] For a 1d particle moving in a potential $V(x)$, what condition does the potential $V(x)$ have to satisfy for parity to be a symmetry (that is, $[H, P_1] = 0$)? Explain. If parity is a symmetry, what are the implications for degeneracy of the eigenstates of H?

D. [**5** points] For the potential $V(x) = k|x|$, is the lowest energy eigenstate an eigenfunction of P_1? If so, what is its parity? Explain briefly.

The angular momentum operator in 3d is given by $\vec{L} = \vec{x} \times \vec{p}$.

E. [**10** points] Calculate the commutator of $\vec{L}$ with the position operators $\vec{x}$, $[L_i, x_j]$, where i, j run over the 3 Cartesian directions x, y and z. Also calculate the commutator of $\vec{L}$ with $\vec{x}^2$.

F. [**5** points] For a spinless particle moving in a 3d potential $V(\vec{x})$, what condition does the potential $V(\vec{x})$ have to satisfy for angular momentum to correspond to a symmetry (that is $[H, \vec{L}] = 0$)? Explain. If angular momentum commutes with the Hamiltonian, what are the implications for the eigenstates of H?

2 Spin-spin interactions (50 points total)

Two spatially separated trapped atoms, each with total spin $S = 3/2$, interact via a spin-spin interaction, that is the Hamiltonian of the system is given by

$$H = -\frac{\gamma}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2$$

where γ is a positive constant with units of energy.

You may find it helpful to recall the action of the ladder operators on J^2, J_z eigenstates:

$$J_{\pm}|j,m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}|j, m \pm 1\rangle.$$

A. [**10** points] There are two bases we can use in order to express any vector in the Hilbert space for the two atoms' spins. We can chose our basis vectors to be $|m_1, m_2\rangle$, denoting eigenvectors of the z-component of atom 1 and 2 with eigenvalue $m_1\hbar$ and $m_2\hbar$ respectively. Another set of basis vectors is $|s, m\rangle$, that is eigenvectors of total spin S^2 and z-component of total spin with eigenvalues $\hbar^2 s(s+1)$ and $m\hbar$ respectively. What is the allowed range of the quantum number m_1, m_2, s and m? What is the dimension of the Hilbert space spanned by these two alternate bases? Explain.

B. [**10** points] Show that the Hamiltonian can be rewritten as

$$H = -\frac{\gamma}{2\hbar^2}\left(S^2 - S_1^2 - S_2^2\right).$$

Show that H is diagonal in one of the two bases described in part a. Which one?

C. [**5** points] Give an example of a perturbation we could add to the Hamiltonian to get a situation where H is diagonal in neither of the two bases? Explain. How could such a perturbation be implemented experimentally?

D. [**25** points] At time $t = 0$ the system is prepared in the $|j = 2, m = 1\rangle$ state. In this state the individual z-components of spin 1 and spin 2 are measured. What are the allowed outcomes of the measurement of m_1 and m_2? Calculate the probabilities for each allowed outcome.

1 Electrostatics (50 points total)

You may find the following equations to be relevant:

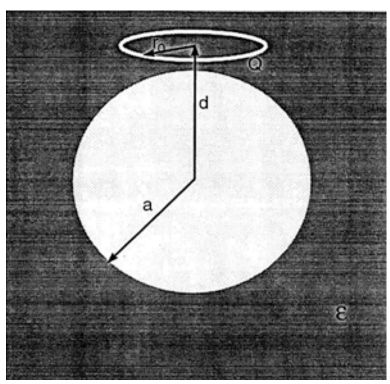
$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y^*_{lm}(\widehat{\mathbf{r}}')Y_{lm}(\widehat{\mathbf{r}})\frac{r^l_<}{r^{l+1}_>},\ Y_{l0}(\theta,\phi) = \delta_{m0}\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$$

A. [**15** points] A ring of charge Q, of radius r_0, is centered at a distance $z = d$ above the origin and lies parallel to the xy plane. Determine the scalar potential $\Phi(\mathbf{r})$ as a series in Legendre polynomials.

B. [**11** points] A Green's function is given by $G(\mathbf{r},\mathbf{r}') = \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{a}{r'}\frac{1}{|\mathbf{r}-\mathbf{r}''|}\right)$, with $\mathbf{r}'' = a^2\mathbf{r}'/|\mathbf{r}'|^2$. Define the properties that a Green's function for electrostatics must have for a Dirichlet boundary value problem and explain why the given $G(\mathbf{r},\mathbf{r}')$ satisfies those properties for a sphere of radius a.

C. [**12** points] A grounded conducting sphere of radius $a < d$ is placed at the origin and embedded in a dielectric of infinite extent with dielectric constant ε. Derive the boundary conditions satisfied by the electrostatic field at the surface of the dielectric from Maxwell's equations.

D. [**12** points] Now consider the ring of part **A** a distance $d > a$ above the center of a conducting sphere as in the figure. Express $\Phi(\mathbf{r})$ in the dielectric medium resulting from the loop-sphere system as series involving Legendre polynomials.



2 Maxwell's Equations and Radiation (50 points total)

A. [**10** points] Write Maxwell's equations for $\mathbf{D}, \mathbf{E}, \mathbf{H}$ and $\mathbf{B}$ in the presence of a free charge density ρ and free current density $\mathbf{J}$. Define $\mathbf{H}$ in terms of the magnetic field $\mathbf{B}$, the magnetization $\mathbf{M}$, and the permeability of free space.

B. [**10** points] Derive equations for the scalar and vector potentials $\Phi(\mathbf{r}, t)$, $\mathbf{A}(\mathbf{r}, t)$ in terms of ρ and $\mathbf{J}$. Assume that the magnetization $\mathbf{M}$, and the polarization $\mathbf{P}$ vanish.

C. [**20** points] A current distribution with a time-dependence $\mathbf{J}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r}) \cos \omega t$ is confined to a region of space of size l. Derive an expression for the electric field $\mathbf{E}(\mathbf{r}, t)$, valid for distances $r \gg l$, in terms of the three-dimensional Fourier transform of $\mathbf{J}(\mathbf{r})$. Determine the time-averaged radiated power per unit solid angle in terms of the same Fourier transform.

D. [**10** points] Consider the situation as in the previous question, but now $\omega r/c \ll 1$. Derive an expression for the vector potential $\mathbf{A}(\mathbf{r})$ in the Lorentz gauge for regions outside a localized distributions of charge and currents.

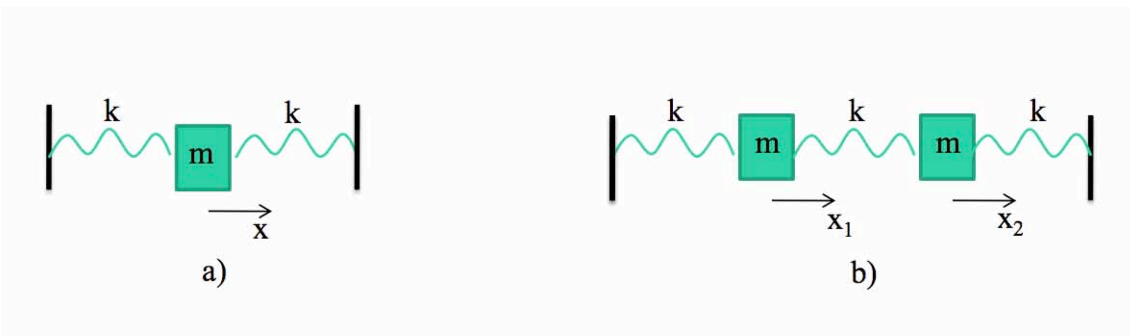
1 Springs and Masses (50 points total)

Treat the following system as obeying classical Newtonian mechanics and in thermal equilibrium with a thermostat at a temperature T. A particle of mass m is attached to two vertical walls by means of springs with an identical spring constant k, as shown in Fig.1a. The mass can move only in the horizontal direction.

You may find the following n-dimensional Gaussian integral identity useful:

$$\frac{\sqrt{\beta}}{(2\pi)^{n/2}} \int x_k x_l \, e^{-\frac{1}{2}\sum_{ij}\beta_{ij}x_i x_j} \, dx_1 dx_2 ....dx_n = \beta^{-1}_{kl} \tag{1}$$

where β is the determinant of matrix β_{ij}, β^{-1}_{ij} is the matrix inverse to β_{ij}, and $i, j = 1, 2...n$.



A. [**10** points] Let x be the displacement of the mass from its equilibrium position. How does the probability to find the mass at a specific displacement x depend on the value of its mass m?

B. [**10** points] Calculate the variance $\langle x^2 \rangle$ in the amplitude of the displacement.

C. [**10** points] Does the correlation function $\langle x(t)x(t') \rangle$ depend on m? Give only a qualitative explanation of your answer.

D. [**5** points] State the content of the equipartition theorem and give one explicit example for when it is invalid.

E. [**5** points] Using (C) calculate the variance of the velocity $\langle v_x^2 \rangle$.

F. [**10** points] Consider now the system of two masses connected by three identical springs, as it is shown in Fig.1b. Calculate the cross correlator $\langle x_1 x_2 \rangle$.

2 High temperature relativistic electrons (50 points total)

Consider a gas of N non-interacting relativistic electrons, $\epsilon_p = cp$, in a three dimensional volume V.

A. [**5** points] Write down the expression for the Fermi-Dirac distribution of the occupation number of electrons $n_{\mathbf{p}}$ in terms of the chemical potential μ, temperature T, and the energy $\epsilon_{\mathbf{p}}$.

B. [**5** points] Write down a formal expression for the total number of electrons in the system involving the density of states using your answer in (A).

C. [**10** points] Consider an experimental setup with a fixed number of electrons and consider very high temperatures where your expression in (B) simplifies. Does the chemical potential μ increase or decrease when the temperature increases?

D. [**10** points] At these very high temperatures the electron gas behaves classically. Write down a formal statistical definition of the Helmholtz free energy F and evaluate it. Show it takes the form

$$F = -Nk_BT \log AT^3 \tag{2}$$

Here A is a constant independent of T. Do not evaluate the value of A.

E. [**10** points] Calculate the heat capacity C_V of the electrons at these high temperatures and compare it with what it would be for non-relativistic electrons.

F. [**10** points] Calculate the magnetic susceptibility density of the electrons χ at these high temperatures in terms of the Bohr magneton β, the density of the electrons N/V and T. Take into account only the spin (Pauli) part of the susceptibility and neglect the spin-orbit interaction.

1 A particle in a central potential (60 points total)

A particle of mass $m = 1$ moves in a potential

$$V = -\frac{1}{r^2} \tag{1}$$

where r is the distance to the center.

A. [**10** points] Write down the Lagrangian of the system, using the polar coordinates r and ϕ.

B. [**10** points] The particle moves from $r = \infty$ along an orbit with angular momentum M. Show that if $M < \sqrt{2}$ then the particle falls into the center of force (capture), and if $M > \sqrt{2}$ it will escape to infinity.

C. [**10** points] Suppose the velocity of the particle at $t = -\infty$ is v_0. Write the condition for capture in terms of the impact parameter b.

D. [**10** points] Express $\dot{r}$ and $\dot{\phi}$ in terms of E, M, and r, and find a differential equation for the trajectory of the particle, $\phi(r)$.

E. [**10** points] Find the trajectory $r(\phi)$ of the particle parameterized by initial angle ϕ_0 in the case $M^2 = 2$.

F. [**10** points] How many rotations does the trajectory make before approaching $r = 0$?

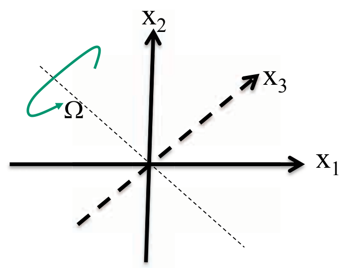


Figure 1:

2 Spinning top (40 points total)

Suppose that the top rotates with a constant angular velocity Ω about the axis shown in Fig.1. by the thin dashed line, which lies in the x_1, x_2 plane at 45^0 angle to the x_1 axis, and goes through the center of mass of the top.

A. [**10** points] Find the body frame components τ_1, τ_2, τ_3 of the torque which should be applied to keep the axis of rotation fixed.

You may use Euler's equations for the angular velocity components in the body frame, ω_1, ω_2 and ω_3.

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + \tau_1 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_1\omega_3 + \tau_2 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_2\omega_1 + \tau_3 \end{aligned} \tag{2}$$

B. [**10** points] Consider now force free motions of the top about each of the axes x_1, x_2 and x_3 with angular velocities ω_1, ω_2 and ω_3 respectively. Which of these three motions are stable and which are unstable with respect to small perturbations?

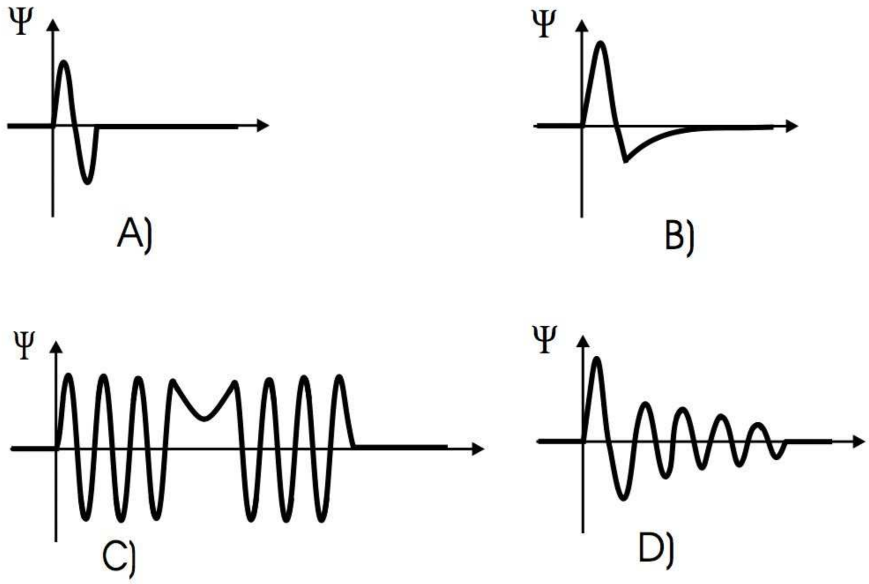
Then consider the case of a symmetric top where $I_1 = I_2$ and figure out which of the motions are stable and which are unstable.

C. [**10** points] Let us choose a stable axis x_i, and assume that the top rotates about it with an angular velocity ω_i. In what direction should a small oscillating torque be applied to induce a *linear* resonance? At what frequency of the torque oscillation Ω does it take place? At what frequencies will *nonlinear* resonances appear?

D. [**10** points] Now consider a force-free rotation about the stable axis x_i, and assume that the corresponding moment of inertia associated with this axis is changing in time as $I_i = I_i^{(0)} + \epsilon \cos \Omega t$, where $\epsilon \to 0$. Identify a sequence of frequencies $\Omega = \Omega_n$ at which *linear* parametric resonances take place. How many such frequencies exist?

1 One Dimensional Quantum Mechanics (50 points total)

A. [**20** points] For each case shown in the figure below, make a sketch of a real (not complex) potential that would generate the stationary wave function shown. Explain your answer in each case. If you believe there is no such potential, explain why.



B. [**15** points] Show that there is always at least one bound state for an attractive, non-singular potential in one dimension. An attractive potential means here one for which $V(x) \leq 0$ for all x, and $V \to 0$ as $x \to \pm\infty$. A bound state then means a state with energy less than zero. You can assume that $\int dx V(x)$ is finite and that V has a finite range, that it is vanishes for $x > R$ for some positive R.

C. [**15** points] A free particle has the initial wave function

$$\psi(x, t = 0) = Ae^{-a|x|},$$

where A and a are positive real constants. Normalize $\psi(x, t = 0)$, then find $\psi(x, t)$ in the form of a single integral.

2 Angular Momentum(50 points total)

Consider a system with 2 particles whose position is frozen. Particle A has spin 3/2, particle B has spin 1.

A. [**10** points] First consider the system in a strong, constant background magnetic field in the z-direction, so that the spin/spin interaction between the two particles can be neglected. The Hamiltonian for this system is given by

$$H = -\mu \left(\vec{S}_A - \vec{S}_B \right) \cdot \vec{B},$$

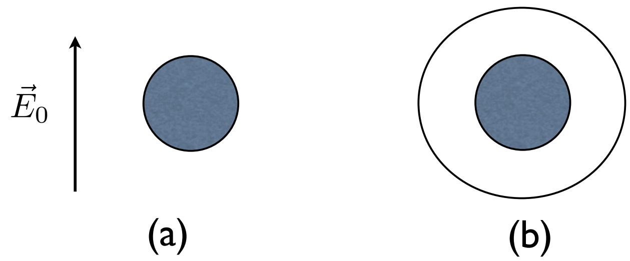
where μ is a positive constant. Find and list the energy eigenstates of this system.

B. [**25** points] The system is initially prepared in a state such that the total spin is 3/2 and the z-component of total spin is 3/2.

- **i.** Determine the probability that the system is measured to have total spin equal to 3/2 as a function of time. If the probability is constant in time, state so explicitly.
- **ii.** Determine the probability that the system is measured to have z-component of total spin equal to 3/2 as a function of time. If the probability is constant in time, state so explicitly.

C. [**15** points] The system is prepared in a different state such that the total spin is 3/2 and the z-component of total spin is 1/2. If the **x-component** of total spin is measured, determine all possible results of the measurement and the probability associated with each.

1 3 D Electrostatics (50 points total)



A. [**15** points] An isolated, uncharged conducting sphere of radius R is placed in a uniform electric field with magnitude E_0 pointing in the z-direction, as shown in the figure (a). Determine the electric potential outside the sphere.

B. [**5** points] Determine the surface charge density on the sphere.

C. [**10** points] Now consider a grounded, conducting sphere of radius R that is surrounded by a ring of charge Q of radius $R_1 > R$, as shown via a view from the top in Fig. (b). The centers of the ring and the sphere are in the same place. A set of Green's functions is given by

$$G(\mathbf{r}, \mathbf{r}') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi Y_{lm}^*(\widehat{\mathbf{r}}') Y_{lm}(\widehat{\mathbf{r}})}{2l+1} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}} \right) \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right),$$

$$\text{and you are given } Y_{l0}(\theta, \phi) = \delta_{m0} \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta). \tag{1}$$

Define the properties that a Greens function for electrostatics must have for a Dirichlet boundary value problem. Define the geometric situation that the given $G(\mathbf{r}, \mathbf{r}')$ is used for. What are the values of a, b for the current situation?

D. [**5** points] Express the charge density $\rho(\mathbf{r})$ in terms of delta functions and known quantities.

E. [**15** points] Determine $\Phi(\mathbf{r})$ for the sphere and ring in the region outside the sphere as a series involving Legendre polynomials.

2 Maxwell's Equations and Wave Propagation (50 points total)

A. [**10** points] Write Maxwell's equations for $\mathbf{D}, \mathbf{E}, \mathbf{H}$ and $\mathbf{B}$ in the presence of a free charge density ρ and free current density $\mathbf{J}$. Define $\mathbf{H}$ in terms of the magnetic field $\mathbf{B}$ the magnetization $\mathbf{M}$, and the permeability of free space. Define $\mathbf{D}$ in terms of the electric field $\mathbf{E}$, the polarization $\mathbf{P}$ and the permeability of free space.

B. [**10** points] Consider a monochromatic plane wave of frequency ω propagating in a medium with constant ϵ and μ. There are no free charges and currents. Consider the proposed solutions of Maxwell's equations: $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$ and $\mathbf{H}(\mathbf{r}, t) = \mathbf{H}\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$. Show that the magnetic and electric fields must be perpendicular to $\mathbf{k}$ and determine the value of n appearing in $k = n\omega/c$.

C. [**20** points] Now consider a monochromatic plane wave of frequency ω propagating in a crystal in which the Cartesian components of $\mathbf{D}$, $\mathbf{E}$ are related by $D_i = \epsilon_{ij}E_j$ in which ϵ_{ij} is given by the matrix $\begin{pmatrix} \epsilon_\perp & 0 & 0 \\ 0 & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{pmatrix}$. There are no free charges and currents and $\mathbf{M} = \mathbf{0}$. The solutions take the same form as in part **B**, with $\mathbf{k} = \omega/c\mathbf{n}$. Show that

(i) $\mathbf{k}, \mathbf{D}$ and $\mathbf{B}$ are mutually perpendicular,

(ii) $\mathbf{k}, \mathbf{D}$ and $\mathbf{E}$ are coplanar,

(iii) $\mathbf{E}\cdot\mathbf{k} \neq 0$,

(iv) Determine the Poynting vector and show that it is not parallel to $\mathbf{n}$.

D. [**10** points] Show that the value of the index of refraction depends on the direction of the wave propagation. This feature leads to the phenomena of birefrigence.

Physics 524: Statistical Mechanics **December 8, 2014, 10:30am-12:20am**
Authumn 2014

Master's Review Exam **Time Limit: 110 Minutes**
Instructor: Chris Laumann

- Please do not turn this page until the buzzer goes at 10:30a.
- This exam contains 3 pages (including this cover page) and 2 problems.
- This is a closed book exam. **No books, notes or calculators allowed.**
- Write all your work on the provided sheets.
- **Organize your work** in a reasonably neat and coherent way.
- **Mysterious or unsupported answers will not receive full credit**.

His colour changed though, when, without a pause, it came on through the heavy door, and passed into the room before his eyes. Upon its coming in, the dying flame leaped up, as though it cried, 'I know him; Marley's Ghost!' and fell again.

The same face: the very same. Marley in his pigtail, usual waistcoat, tights and boots; the tassels on the latter bristling, like his pigtail, and his coat-skirts, and the hair upon his head. The chain he drew was clasped about his middle. It was long, and wound about him like a tail; and it was made (for Scrooge observed it closely) of cash-boxes, keys, padlocks, ledgers, deeds, and heavy purses wrought in steel. His body was transparent; so that Scrooge, observing him, and looking through his waistcoat, could see the two buttons on his coat behind.

Scrooge had often heard it said that Marley had no bowels, but he had never believed it until now.

— Charles Dickens, A Christmas Carol (1897)

1. A solid in 1D is composed of N atoms of mass m equally spaced on a line with equilibrium lattice constant a. Within the harmonic approximation, the atoms may be viewed as being attached to their neighbors by tiny springs with spring constant K. Thus, the microscopic Hamiltonian is

$$H = \sum_i \frac{p_i^2}{2m} + \sum_i \frac{1}{2} K (u_i - u_{i+1})^2 \tag{1}$$

where u_i is the displacement of the i'th atom relative to its equilibrium position $x_i = ia + u_i$, and p_i is the conjugate momentum.

   (a) (5 points) Assuming the atoms are classical, state the internal energy $E(T)$ as a function of temperature T. Briefly justify your answer.

   (b) (5 points) What is the corresponding classical heat capacity C of the solid?

   (c) (10 points) Normal mode coordinates may be found by Fourier transformation $u_j = \frac{1}{\sqrt{N}} \sum_k e^{ikj} \tilde{u}_k$. Calculate the dispersion relation for the normal modes $\omega(k)$ and sketch it in the Brillouin zone.

   From here, let us assume a Debye model for the phonon spectrum, $\omega(k) = v|k|$ where v is the speed of sound.

   (d) (5 points) After quantization, what is the expected number of phonons in the mode at wavevector k?

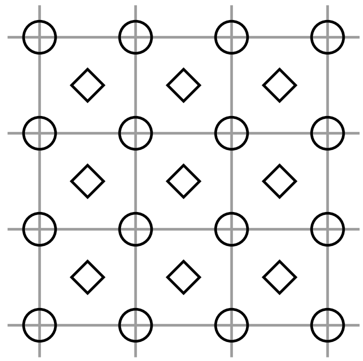
   (e) (15 points) Sketch the quantum heat capacity $C(T)$ of the solid and derive its leading behavior at low and high temperatures. What is the characteristic temperature T_d separating these limits? You need not evaluate any *dimensionless* integrals.

   The Lindemann criterion holds that a solid melts when the thermal fluctuations of the position of an atom become larger than the spacing between atoms.

   (f) (15 points) What is the variance of the position of the i'th atom $\langle u_i^2 \rangle$ at temperature T? You may assume the atoms are classical.

   (g) (5 points) Is the 1-D solid stable to thermal fluctuations?

2. A two-dimensional monatomic crystal consists of a very large number N of atoms. In a perfect crystal, the atoms sit on the sites of a square lattice (circles in the Figure). However, at a cost of energy ϵ, an atom may instead sit on an interstitial site at the center of a face of the lattice (diamonds in the Figure).



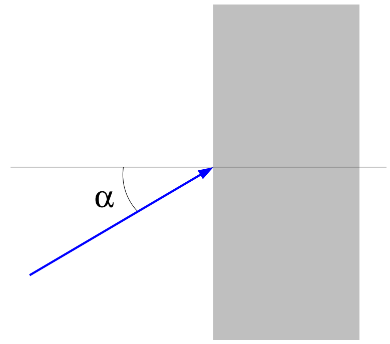
   (a) (15 points) Calculate the entropy for states with n atoms on interstitial sites assuming $1 \ll n \ll N$ and that any atom originally on a circle site may only hop to one of the four nearest interstitial sites. Neglect interactions between atoms on different sites.

   (b) (15 points) Repeat the calculation in (a), assuming that the n unoccupied sites on the circle lattice are uncorrelated with the n occupied interstitial sites. (Still $1 \ll n \ll N$).

   (c) (10 points) What is the fraction of interstitial sites that are occupied at low temperatures $k_b T \ll \epsilon$ for the two cases (a) and (b) above?

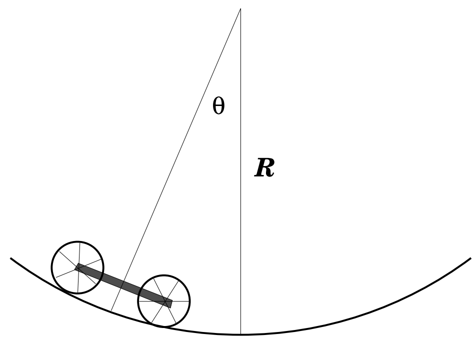
1 Particle Refraction

[**20** points] A particle of mass m and energy E impinges on a potential step with positive heigth V_0: $V(\vec{x}) = \begin{cases} 0\,, & x_3 < 0; \\ V_0\,, & x_3 > 0. \end{cases}$ For a given angle of incidence α, what is the *minimum* energy E_0 for which the particle is not reflected by the potential step?



2 Rolling Cart

[**25** points total] A four-wheeled cart rolls, without slipping, on a cylindrical surface with hemispherical cross-section. The radius of curvature of the surface is R. The cart has a rectangular bed of length 4ℓ, mass M, and negligible height. Attached to the bed of the cart are four wheels of radius ℓ and mass m. The mass of the wheels is concentrated in the rims; you may neglect the mass of the spokes and axles. Gravity acts downward.

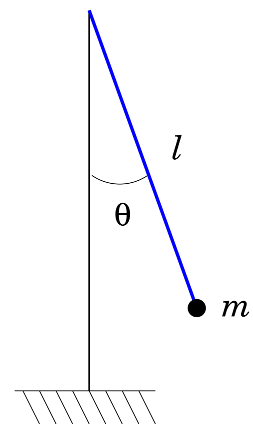


A. [**20** points] Introduce appropriate generalized coordinates and write the Lagrangian for this system. Simply the result.

B. [**5** points] What is the frequency of small oscillations of the cart about the bottom of the surface?

(continued on next page)

3 Tetherball



[**55** points total] A ball of mass m hangs from a rope of length ℓ attached to a swivel at the top of a thin rigid vertical pole. The size of the ball is small compared to ℓ and may be neglected. The rope is inextensible. The mass of the rope and swivel are negligible compared to the ball. Friction in the swivel is negligible. Gravity acts downward.

Let θ denote the polar angle between the rope and the pole, and ϕ the azimuthal angle around the pole. A person holding the ball at some angle θ_0, with the rope fully extended, strikes and simultaneously releases the ball imparting an initial velocity to the ball which is purely in the azimuthal direction. The initial strike gives the ball some non-zero angular momentum (about the vertical), L_z.

A. [**9** points] Write the Lagrangian for the system, assuming that the rope remains fully extended and always has positive tension. Define the zero of the gravitational potential to lie at the top of the pole. What are the resulting equations of motion? What is L_z, and the total energy E, in terms of your generalized coordinates and velocities?

B. [**6** points] The equations of motion may be reduced to a decoupled equation for $\theta(t)$ corresponding to motion in some effective potential $V_{\text{eff}}(\theta)$. What is $V_{\text{eff}}(\theta)$ and what is the resulting equation of motion for θ?

C. [**20** points] Explain why the trajectory of the ball will be confined to an interval in θ bounded by θ_0 and some other angle θ_1, with $0 < \theta_1 < \pi$. What determines θ_1? Use this relation to express L_z^2 in terms of θ_0 and θ_1, and simplify the result.

D. [**20** points] What is the tension T in the rope? Suppose $\theta_0 = \pi/12$ and $\theta_1 = 4\pi/3$. Does the rope ever go slack?

1 Non-relativistic Hydrogen atom (50 points total)

Consider the non-relativistic hydrogen atom where we treat the proton as being infinitely massive and point-like, so that the electron moves in a stationary Coulomb potential. The effects of spin may be ignored. The system is prepared at a time $t = 0$ in the normalized state

$$\psi(\mathbf{r}, t = 0) = \frac{1}{\sqrt{6}} \left[\phi_{31-1}(\mathbf{r}) + 2\phi_{310}(\mathbf{r}) + \phi_{42-1}(\mathbf{r}) \right], \quad (1)$$

where $\phi_{nlm}(\mathbf{r}) = R_{nl}(r) Y_{lm}(\theta, \phi)$ are normalized energy eigenfunctions with energy eigenvalues

$$E_n = -\frac{\mathrm{R_y}}{n^2}, \quad \mathrm{R_y} = \frac{e^2}{2a_0} = 12.6\,\mathrm{eV}.$$

A. [**5** points] Does the state $\psi(\mathbf{r}, t = 0)$ have definite parity? If so, determine its parity. Explain your reasoning.

B. [**5** points] Determine the expectation value of the energy at time $t = 0$. Express your answer in terms of $\mathrm{R_y}$.

C. [**5** points] Compute $\mathbf{L}^2 \psi(\mathbf{r}, t = 0)$, where $\mathbf{L}$ is the orbital angular momentum operator.

D. [**10** points] Compute the expectation value of L_z at $t = 0$.

E. [**10** points] Compute the expectation value of L_x at a time $t > 0$. Hint:

$$(L_x \pm i\, L_y) Y_{lm} = \hbar \sqrt{(l \mp m)(l \pm m + 1)} Y_{lm\pm 1}.$$

F. [**15** points] Now consider a different time-independent situation. The system with wavefunction given by Eq. (1) is also exposed to a very weak electric field of magnitude E_0 in the positive z direction. Compute the change in energy. Express your answer in terms of well-defined, non-vanishing integrals.

2 Approximation methods (50 points total)

A. [**25** points] Two particles, of reduced mass m, moving in three dimensions interact via a Yukawa potential $V(r) = -\frac{\hbar^2}{2m}\lambda\mu\frac{e^{-\mu r}}{r}$, with λ, μ both real and positive. Furthermore, $\lambda > 2$. Show that a bound state (with energy < 0) must exist (20) and determine a lower bound (5) for the ground state energy. The following expression may be helpful:

$$\int_0^\infty dr\, r^n e^{-\gamma r} = \frac{n!}{\gamma^{n+1}}$$

B. [**25** points] The $1s$ wave function of the hydrogen atom is given by

$$\psi_{1s}(\mathbf{r}) = \frac{e^{-r/a_0}}{\sqrt{\pi\, a_0^3}},$$

where $a_0 = \frac{\hbar^2}{me^2}$ is the Bohr radius. Find the shift in the energy of the $1s$ level due to the finite size of the proton using lowest-order perturbation theory. You may assume that the proton charge distribution $\rho_p(r)$ is spherically symmetric and non-vanishing only within a size $\ll a_0$. Express your answer in terms of $r_p^2 \equiv \int d^3r\, r^2\, \rho_p(r)$. The following expression may be helpful:

$$\frac{1}{4\pi r} = \frac{1}{(2\pi)^3}\int \frac{d^3q}{q^2} e^{-i\mathbf{q}\cdot\mathbf{r}}.$$

1 An Anisotropic Conductor (50 points total)

Consider a conductor in which the free current density and the electric displacement are related linearly via

$$\vec{J}_{free} \;=\; g\left(\hat{z}\times\vec{D}\right)\ ,$$

where g is a real, positive constant. (The Hall effect is represented by a current of this form, with g proportional to the Hall conductivity.) Assume that the conductor's polarization and magnetization vectors are zero ($\vec{P}=0$, $\vec{M}=0$). A monochromatic plane wave travels through such a conductor in the $+\hat{z}$ direction:

$$\begin{aligned}\vec{E}(z,t) &= Re[\vec{E}_0 e^{i(kz-\omega t)}]\ ;\\ \vec{B}(z,t) &= Re[\vec{B}_0 e^{i(kz-\omega t)}]\ ,\end{aligned}$$

where ω is the given (real) angular frequency of the wave. $\vec{E}_0$, $\vec{B}_0$ and k are constants to be determined.

A. Show, using the Maxwell equations, that one can write

$$X_{ij}E_j \;=\; 0\ ,$$

where E_j are the cartesian components of the electric field, the summation convention is assumed, and

$$\hat{X} \;=\; \begin{pmatrix} n^2-1 & ig/\omega & 0\\ -ig/\omega & n^2-1 & 0\\ 0 & 0 & -1\end{pmatrix}\ . \tag{1}$$

[**18** points]

B. Find the possible values for the index of refraction n for this wave. Express your answers in terms of g and ω. [**12** points]

C. Show that there is a cut-off frequency ω_c such that for $\omega>\omega_c$, all the waves which have definite n propagate without attenuation. Show that for $\omega<\omega_c$, only one of these waves propagates and the other wave does not propagate but undergoes attenuation. Find the value for ω_c and explain your reasoning. [**10** points]

D. For each possible value of n, find the corresponding electric-field polarization $\vec{E}_0$. [**10** points]

A formula that may prove useful:

$$det(\hat{A}_{3\times 3}) \;=\; \epsilon_{ijk}A_{1i}A_{2j}A_{3k}$$

2 An Oscillating Dipole (50 points total)

An oscillating electric dipole moment,

$$\vec{p}(t) \quad = \quad p_0 \cos(\omega t)\,\hat{z}$$

generates radiating electric and magnetic fields. p_0 and the angular frequency ω are given positive real constants.

A. Write down a time-dependent charge-density distribution that gives rise to this oscillating dipole moment. [**10** points]

B. Show that far away from the dipole, in the radiation zone ($r \to \infty$), the vector potential may be given by

$$\vec{A}(t) \quad = \quad -\frac{\mu_0 p_0 \omega}{4\pi r}\hat{z}\sin[\omega(t - r/c)] \ .$$

Be sure to explain every step. [**10** points]

C. Calculate the magnetic field due to this dipole and express your result in spherical coordinates. Do all terms in your expression correspond to propagating radiation? [**10** points]

D. How is the electric field oriented relative to the magnetic field? [**5** points]

E. Find the (time-averaged) Poynting vector $\vec{S}$. [**5** points]

F. Find the time-averaged power detected per unit solid angle, $dP/d\Omega$, in the (θ, ϕ) direction. [**5** points]

G. Find the total power of radiation emitted from this dipole. [**5** points]

An integral that may prove useful:

$$\int_0^\pi \sin^3\theta d\theta \quad = \quad \frac{4}{3}$$

1 Energy fluctuations (20 points total)

A. [**10** points]Starting from the Gibbs distribution obtain the following expressions for the average energy of a system, $\langle E\rangle$, and its variance, $\langle \delta E^2\rangle$:

$$\langle E\rangle = -\frac{\partial \ln Z}{\partial \beta}, \quad \langle \delta E^2\rangle = \frac{\partial^2 \ln Z}{\partial \beta^2}.$$

Here $\beta = \frac{1}{T}$ is the inverse temperature and $Z = \sum_n \exp(-\beta\varepsilon_n)$ is the partition function of the system.

B. [**10** points]Express the variance of energy fluctuations in terms of the heat capacity of the system at constant volume, C_V.

2 Boltzmann gas (25 points total)

Consider a gas of N neutral ^{3}He atoms in a container of volume V at temperature T. Assuming the gas can be treated as ideal and classical find the following.

Note: The nuclear spin of ^{3}He is 1/2. The total electron spin is 0.

A. [**15** points] The partition function Z for the gas. Show your work (briefly).

B. [**5** points] The Helmholtz free energy F.

C. [**5** points] The chemical potential μ.

3 Bose-Einstein condensation (30 points total)

Find the ratio $\frac{TS}{E}$, where S, T and E are respectively the entropy, temperature and energy, for an ideal three dimensional non-relativistic Bose gas below the BEC temperature.

4 Phase equilibrium (25 points total)

Consider a liquid (consisting of a single substance) in equilibrium with its vapor.

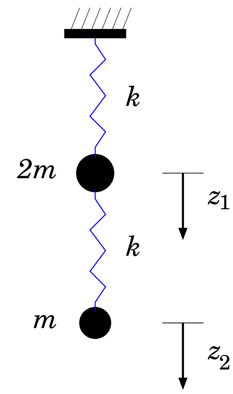
A. [**10** points] State the condition of the phase equilibrium, and derive the Clapeyron-Clausius equation for the slope, $\frac{dP}{dT}$, of the phase equilibrium line.

B. [**15** points] Consider heating of the vapor along the phase equilibrium line. Find the specific heat per molecule of the vapor for this process. Express the result in terms of the specific heat at constant pressure, c_P and the latent heat q per molecule. Assume that the density of the vapor is much smaller than that of the liquid, and that the vapor may be treated as an ideal gas.

This exam consists of Problem 1 (with parts A-B), Problem 2 (with parts A-E), and Problem 3 (with cases A-C). Write your solutions for each problem on the empty pages following that problem. There are 150 points in the exam.

1 Hanging Masses [**50** points total]

Two masses, $m_1 = 2m$ and $m_2 = m$, are suspended in a uniform gravitational field g by identical massless springs with spring constant k. Assume that only vertical motion occurs, and let z_1 and z_2 denote the vertical displacement of the masses from their *equilibrium* positions.

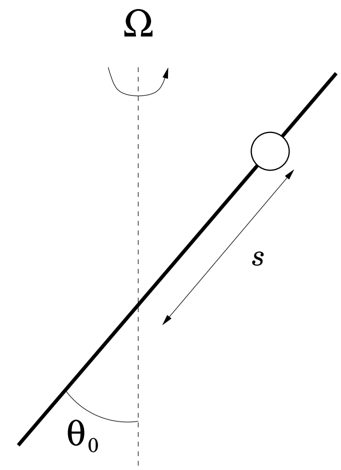


A. [**10** points] Construct the Lagrangian and find the resulting equations of motion.

B. [**40** points] Solve for the subsequent motion given initial conditions $z_2 = \delta h$, and $z_1 = \dot{z}_1 = \dot{z}_2 = 0$ at time $t = 0$.

2 Bead on Wire [70 points total]

A bead of mass m slides without friction on a rigid straight wire which is inclined at a fixed angle θ_0 from the vertical and rotates at a constant angular velocity Ω about a vertical axis. Gravity acts downward.



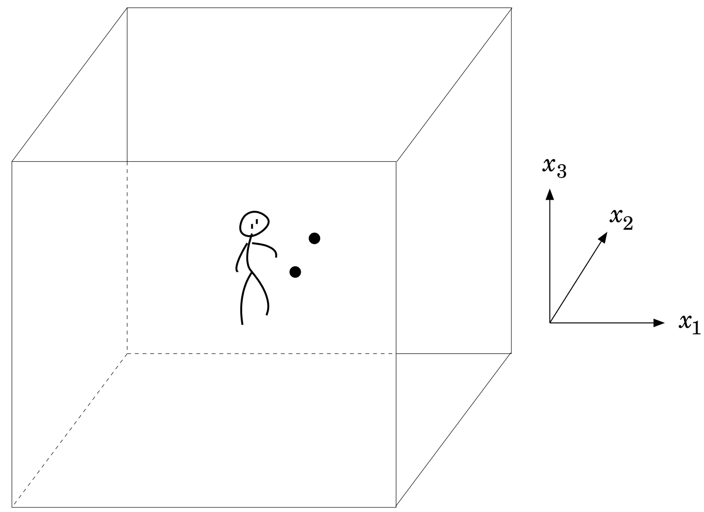
A. [**10** points] Let s denote the displacement along the wire of the bead, relative to the point on the wire where it intersects the axis of rotation. Construct the Lagrangian and derive the resulting equations of motion.

B. [**10** points] Find the general solution to the equations of motion. Show that there is an equilibrium point for the bead at some position s_0. Is this a point of stable or unstable equilibrium?

C. [**35** points] Find the constraint force acting on the bead (i.e., the force the wire exerts on the bead), expressed in terms of s and $\dot{s}$. At what rate does the constraint force do work on the bead?

D. [**15** points] Construct the Hamiltonian of the system, and explain the physical meaning of each term appearing in H. Is the Hamiltonian conserved? Is the kinetic plus potential energy of the bead conserved?

3 Geodesic deviation [30 points total]



An astronaut floats in a space station in a circular orbit about the Earth, at rest inside a windowless room, in the center of which is the center of mass of the station. At all times, the leading direction of the orbit is the direction labeled $\hat{x}_1$ and the Earth lies in the $-\hat{x}_3$ direction. The station has a known orbital period of T.

The astronaut has two identical marbles, and a clock, and performs a series of experiments in which she releases both marbles with some initial separation and negligible initial velocity (relative to the walls of the room). For the purpose of this problem, assume that Earth's gravity is the only relevant force acting on the marbles.

Clearly describe, as fully as possible, what will be observed:

A. [**10** points] when the initial separation of the marbles is in the $\hat{x}_1$ direction;

B. [**10** points] when the initial separation of the marbles is in the $\hat{x}_2$ direction;

C. [**10** points] when the initial separation of the marbles is in the $\hat{x}_3$ direction.

This exam consists of two parts, Problem 1 (with sections A-E) and Problem 2 (with sections A-D). Write your solutions for Problem 1 on the empty pages in-between and for Problem 2 on the empty pages at the end.

1 Non-Relativistic Hydrogenic atoms (50 points total)

Consider the non-relativistic hydrogen atom where we treat the proton as being infinitely massive and point-like, so that the electron moves in a stationary Coulomb potential. The effects of spin may be ignored. The system is prepared at a time $t = 0$ in the normalized state

$$\psi(\mathbf{r}, t = 0) = \frac{1}{\sqrt{7}} \left[2\phi_{100}(\mathbf{r}) + \phi_{311}(\mathbf{r}) + \sqrt{2}\phi_{211}(\mathbf{r}) \right], \quad (1)$$

where $\phi_{nlm}(\mathbf{r}) = R_{nl}(r)Y_{lm}(\theta, \phi)$ are normalized energy eigenfunctions with energy eigenvalues

$$E_n = -\frac{\mathrm{R_y}}{n^2}, \quad \mathrm{R_y} = \frac{e^2}{2a_0} = 13.6\,\mathrm{eV}.$$

$$(L_x \pm i\, L_y)Y_{lm} = \hbar\sqrt{(l \mp m)(l \pm m + 1)}Y_{lm\pm1}.$$

A. [**10** points] At $t = 0$ determine the probability P_{11} that the state has $l = 1$ and $m = 1$.

B. [**10** points] Consider the time evolution of the given wave function to a time $t_1 > 0$. Does the expectation value of r^2 depend on the value of t_1? Explain.

C. [**10** points] For later times t_1, does P_{11} depend on time? Explain.

D. [**10** points] Consider the time evolution of the given wave function to a time $t_1 > 0$. Determine the overlap of $\psi(\mathbf{r}, t_1)$ with a state that has $l = 1$, and is an eigenstate of L_x with eigenvalue=1.

E. [**10** points] The alkali atoms have an electronic structure which resembles that of hydrogen. In particular, the spectral lines and chemical properties are largely determined by one electron(outside closed shells). A model for the potential in which this electron moves is

$$V(r) = \frac{-e^2}{r}\left(1 + \frac{b}{r}\right),$$

with $b < 0$. Determine the bound energy levels of such a potential.

2 Symmetries and Approximation Methods (50 points total)

A spin 1/2 particle (of charge q) moves in three dimensions. The Hamiltonian is given by

$$H = \frac{p^2}{2m} + V(r) + \lambda \boldsymbol{\sigma} \cdot \mathbf{r}\, h(r).$$

where $V(r), h(r)$ are spherically symmetric functions of the distance from the origin.

A. [**13** points] List all of the compatible observables.

B. [**12** points] Now suppose that $V(r) = \frac{1}{2}m\omega^2 r^2$, $h(r) = 1$ and λ can be regarded as being very small. Compute the energy of the ground state, including terms up to second order in λ.

C. [**12** points] Consider the previous part , but also let the particle be exposed to a very weak, constant external electric field $\mathbf{E_0}$. Compute the ground state energy to non-vanishing lowest order in λ and $\mathbf{E}_0$, and comment on its dependence on the direction of the spin of the particle.

D. [**13** points] Now suppose that $\lambda = 0$, and that the system, initially in its ground state, is exposed to a very weak electric field: $\mathbf{E}_0 \cos \bar{\omega} t$, starting at $t = 0$. Determine the probability that the system is in its first excited state for times $t > 0$.

Possibly useful formulae

$$b = \sqrt{\frac{\hbar}{m\omega}}$$

$$R_{00}(r) = \frac{2}{\pi^{1/4}} \frac{1}{b^{3/2}} e^{-r^2/2b^2}$$

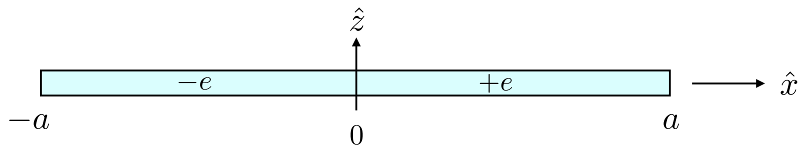
$$R_{01}(r) = \frac{2}{\pi^{1/4}} \frac{1}{b^{3/2}} e^{-r^2/2b^2} \sqrt{\frac{2}{3}} \frac{r}{b}$$

$$R_{10}(r) = \frac{2}{\pi^{1/4}} \frac{1}{b^{3/2}} e^{-r^2/2b^2} \sqrt{\frac{2}{3}} (\frac{3}{2} - \frac{r^2}{b^2})$$

$$\int_0^\infty dr e^{-\lambda r^2} = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}$$

1 An Oscillating Dipole (50 points total)

A thin rod of length $2a$ has a uniform distribution of positive charge $+e$ on one half and negative charge $-e$ on the other half, forming an electric dipole. It lies in the $\hat{x}-\hat{y}$ plane as in the figure below and is set rotating at angular frequency ω about the $\hat{z}$ axis.



A. Determine its time-dependent electric dipole moment $\vec{p}(t)$. Write the dipole moment in complex form with an assumed dependence on $\exp(-i\omega t)$. [**10** points]

B. Recall that the vector potential of the dipole outside the source is given by:

$$\vec{A}(\mathbf{x}) \quad = \quad -\frac{i\mu_0\omega}{4\pi}\,\vec{p}\,\frac{e^{ikr}}{r}\ . \tag{1}$$

From this expression compute the electric and magnetic fields in the radiation zone. [**10** points]

C. Find the formula for the time-averaged power radiated per unit solid angle, $dP/d\Omega$, far from the dipole. Give the result as a function of spherical angles (θ, ϕ) that describe some direction in space, outward from the dipole. [**10** points]

D. Calculate the total power radiated by the dipole. [**10** points]

E. What is the flux of the angular momentum density in the radiation zone? [**10** points]

2 Optically Active Medium (50 points total)

An optically-active medium can rotate the plane of polarization of light by allowing right- and left-circularly polarized waves that obey different dispersion relations. The electric susceptibility tensor of such a medium can be expressed as

$$\hat{\chi} = \begin{pmatrix} \chi_{11} & i\chi_{12} & 0 \\ -i\chi_{12} & \chi_{11} & 0 \\ 0 & 0 & \chi_{33} \end{pmatrix}, \tag{2}$$

where $\hat{\chi}$ is related to the electric polarization $\vec{P}$ in the usual way: $P_i = \epsilon_0 \chi_{ij} E_j$. Note that χ_{11}, χ_{12} and χ_{33} are real constants.

A. Derive the wave equation satisfied by the electric field in this medium. [**10** points]

B. Now assume that a plane wave propagates in the medium in the $\hat{z}$ direction (which is also the 3-direction) with frequency ω. Show that the propagating electromagnetic wave is transverse. [**10** points]

C. Show that the medium admits electromagnetic waves of two distinct wave vectors of magnitude k_R and k_L. Find these wave vectors in terms of ω and the necessary elements of $\hat{\chi}$. [**10** points]

D. Show that the wave vectors, k_R and k_L, correspond to the propagation of right- and left-circularly polarized electromagnetic waves. [**10** points]

E. Find an expression for the difference of the indices of refraction, $n_L - n_R$, *the rotary power*, in terms of the elements of $\hat{\chi}$. [**10** points]

1 Boltzmann gas (35 points total)

Consider a gas of N neutral ^{3}He atoms in a container of volume V at temperature T. Assuming the gas can be treated as ideal and classical find the following quantities:

(Note: The nuclear spin of ^{3}He is 1/2.)

A. [**15** points] The partition function Z for the gas. Show your work (briefly).

B. [**10** points] The chemical potential μ.

C. [**10** points] The entropy per atom $s = \frac{S}{N}$.

2 Dissociation equilibrium (30 points total)

Consider the hydrogen gas at a temperature $T \ll \hbar^2/2I \approx 85\,^\circ\mathrm{K}$, where I is the moment of inertia of the hydrogen molecule. The binding energy of the molecules is E_b.

At sufficiently low density of the molecular gas, $n_m = N_m/V$, some fraction of the molecules is dissociated. Answer the following questions.

A. (10 pts) State the condition of equilibrium with respect to dissociation.

B. (20 pts) Find the density of atomic hydrogen in the gas $n_a = N_a/V$ assuming $n_a \ll n_m$.

What did you assume about the nuclear spin of the hydrogen molecule? Briefly explain your reasoning (5 of 20 pts). (Note: The hydrogen nuclear spin is 1/2.)

3 Ideal Bose gas (35 points total)

A thermally insulated container is separated by a wall into two compartments of volumes V_L and V_R. In the initial state the right compartment is empty. The left compartment is filled with N atoms of an ideal spinless Bose gas at temperature T_i, which is lower than the Bose-Einstein condensation temperature T_0.

A. [**10** points] Express the condensate fraction N_0/N in the initial state in terms of T_i and T_0. Give a brief derivation. You do not need to evaluate the dimensionless integrals that arise in the process.

B. The wall separating the container is removed, and the Bose gas expands onto the rest of the container.

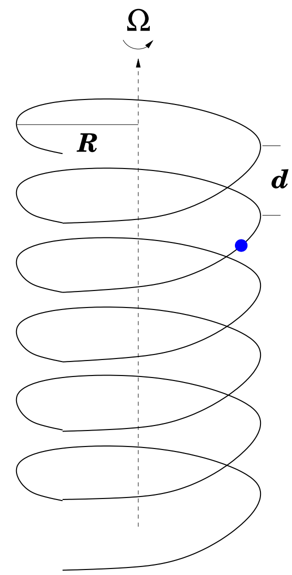
i. [**10** points] Assuming that in the final state the gas remains Bose condensed determine the final temperature T_f.

ii. [**15** points] Find the maximal volume of the right compartment, V_R for which the gas in the final state remains Bose condensed.

This exam consists of Problems 1 and 2 with parts A and B, and Problem 3 with parts A–C. Write your solutions for each problem on the empty pages following that problem. There are 200 points in the exam.

1 Bead on helix

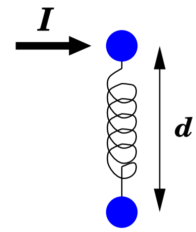
[**50** points total] A bead of mass m slides without friction along a wire bent into a right-handed circular helix whose turns have radius R and pitch d, as shown. The axis of the helix is vertical, and gravity acts downward. The helical wire is rotating at angular velocity Ω about its central axis.



A. [**25** points] Construct the Lagrangian of the system, derive the resulting equations of motion, and find the most general solution of these equations. (Feel free to introduce abbreviations, clearly defined, for relevant combinations of parameters.)

B. [**25** points] Construct the Hamiltonian of the system, and derive the resulting Hamilton's equations. Show that your Hamiltonian equations of motion are equivalent to your Euler-Lagrange equation.

2 Diatomic motion



[**70** points total] Two particles of mass m slide, without friction, on a horizontal plane. The masses are connected by an ideal massless spring with spring constant k which, at the initial time $t_0 = 0$, is stretched to a length of d. The equilibrium length of the spring is small compared to d, and may be neglected. The two masses are initially at rest. At the instant of release, $t = t_0$, a momentary impulse I is delivered to one of the masses, in a direction perpendicular to the initial separation, as shown.

A. [**30** points] Qualitatively describe the subsequent motion. What quantities are conserved and what are their values? Do the two masses ever strike each other?

B. [**40** points] Let $\mathbf{r}(t)$ denote the separation between the two masses. What are the initial conditions: $\mathbf{r}_0 \equiv \lim_{t\to 0^+} \mathbf{r}(t)$ and $\mathbf{v}_0 \equiv \lim_{t\to 0^+} \dot{\mathbf{r}}(t)$? What is $\ddot{\mathbf{r}}(t)$? Solve for $\mathbf{r}(t)$.

3 Trapped?

[**80** points total] A particle of mass m moves in a two dimensional potential $V(x, y) = \frac{1}{2}\lambda\, x^2\, y^2$. At time $t = 0$, the particle is at the origin and is moving in the direction $\hat{n} = (\cos\theta_0, \sin\theta_0)$ with initial kinetic energy E. The initial angle is small but non-zero, $0 < \theta_0 \ll 1$. Your task is to estimate, as accurately as you can, the maximum distance from the origin which the particle can reach. To do so:

A. [**10** points] Explain why one may set $m = \lambda = 2E = 1$, with no loss of generality. Do so for the bulk of this problem.

B. [**25** points] Carefully sketch a contour plot of the potential. Label relevant contour lines. Then draw on your plot what you expect the particle's trajectory, released with, e.g., $\theta_0 \approx 0.25$, will look like. Hint: visualize how a landscape with $V(x, y)$ as the elevation appears if you were standing at the origin.

C. [**45** points] As the particle moves toward increasing values of x, the motion in y will be oscillatory, with decreasing amplitude and increasing frequency. Justify this. A rough approximation to the amplitude of the transverse y oscillations is $A(x) \approx (x^2 + \theta_0^{-2})^{-1/2}$. Justify this by considering both early oscillations with x small compared to θ_0^{-1}, and late oscillations with x larger than θ_0^{-1}. Use this to formulate an effective slow dynamics for x in which the fast oscillations in y are averaged out. Using the resulting effective dynamics in x, solve for the maximal value of x. Write the result for $x_{\max}$ with all dependence on m, λ, E, and θ_0 made explicit.

This exam consists of two parts, Problem 1 (with sections A-E) and Problem 2 (with sections A-D). Write your solutions for Problem 1 on the empty pages in-between and for Problem 2 on the empty pages at the end.

1 The basics (50 points total)

A spin-less particle of mass m moves in a spherically-symmetric, infinitely-deep potential well of radius a. The normalized state at a time $t = 0$ is given by

$$\psi(\mathbf{r}, t = 0) = \frac{1}{\sqrt{35}} \left[\phi_{010}(\mathbf{r}) + 3\phi_{311}(\mathbf{r}) + 5\phi_{221}(\mathbf{r})\right], \quad (1)$$

where $\phi_{nlm}(\mathbf{r}) = R_{nl}(r)Y_{lm}(\theta, \phi)$ are normalized energy eigenfunctions with energy eigenvalues. The radial quantum number n gives the number of nodes for positions $0 < r < a$.

$$(L_x \pm i\, L_y)Y_{lm} = \hbar\sqrt{(l \mp m)(l \pm m + 1)}Y_{lm\pm 1}.$$

Zero's of spherical Bessel functions for node number n and orbital angular momentum l:

$x_{n0} = \{3.14159, 6.28319, 9.42478\}$, $x_{n1} = \{4.49341, 7.72525, 10.9041\}$, $x_{n2} = \{5.76346, 9.09501, 12.3229\}$

A. [**9** points] At $t = 0$ determine the probability that the state has $l = 1$.

B. [**9** points] At $t = 0$ determine the probability density for the particle to be at the center of the well. Explain.

C. [**9** points] At $t = 0$ the energy of the system is measured and the largest possible value is found. Determine the state of the system immediately after the measurement.

D. [**9** points] Now instead suppose that just after $t = 0$ the position of the particle is measured, and then immediately afterwards the energy is measured. What set of values of the energy could you possibly have obtained?

Consider another situation. Two spin-1/2 particles are separated by $\mathbf{a} = a\widehat{z}$, and interact only by a magnetic dipole interaction:

$$H = \mu^2 \left[\frac{\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2}{a^3} - \frac{3\boldsymbol{\sigma}_1 \cdot \mathbf{a}\, \boldsymbol{\sigma}_2 \cdot \mathbf{a}}{a^5}\right].$$

E. [**14** points]. Determine the compatible observables and all of the energy levels.

2 Symmetries and Approximation Methods (50 points total)

A spin 1/2 particle (of charge q and mass m) moves in three dimensions. It is in an eigenstate $|\psi\rangle$ of the Hamiltonian. Suppose that the spatial wavefunction is given by

$$\langle \mathbf{r}|\psi\rangle = N r e^{-\beta r} Y_{1,0}(\theta,\phi).$$

The Hamiltonian is given by $H_0 = \frac{p^2}{2m} + V(r)$ where $V(r)$ is a spherically symmetric function of the distance from the origin.

A. [**13** points] Determine the eigenenergy and the potential $V(r)$.

B. [**12** points] Now include an additional term in the Hamiltonian $V_{LS}(\mathbf{r}) = \frac{C}{r^3}\mathbf{L}\cdot\mathbf{S}$. Determine a condition for the validity of first-order perturbation theory, and use this approximation to determine an expression for the energy shift of the state $|\psi\rangle$. Your answer should be expressed in terms of a well defined spatial integral. You do not need evaluate the integral. Be sure to include the effects of spin and orbital motion on the energy levels.

C. [**13** points] Now suppose that the weak interaction causes an additional very small potential $V_W = \lambda\boldsymbol{\sigma}\cdot\mathbf{r}$ to exist. Assume that you have obtained the wave function correctly to first-order in the parameter λ, and measure the orbital angular momentum, l. What values of l could you obtain? What values of the total angular momentum j would you obtain?

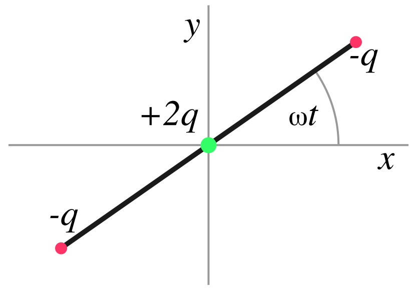
D. [**12** points] Now neglect V_{LS} and V_W. Suppose that the system, initially in the state $|\psi\rangle$, is exposed to a very weak electric field: $\mathbf{E}_0\cos\omega t$, starting at $t = 0$. Determine the probability that the system is in its ground state for times $t > 0$. You may express your answer in terms of well-defined integrals over time and space. You do not need to evaluate the integrals.

Possibly useful formulae

$$\int_0^\infty dr\, r^n e^{-\lambda r} = \frac{n!}{\lambda^{n+1}}.$$

1 A rotating quadrupole (50 points total)

Three charges, $-q$, $-q$, and $+2q$ are fixed at the two ends and center, respectively, of a rod of length $2a$ that rotates in the xy-plane (in vacuum) at angular speed ω around the z-axis. This forms a rotating electric quadrupole as in the figure.



A. Determine the charge density, $\rho(\mathbf{x}, t)$, in Cartesian coordinates. [**5** points]

B. Determine the non-vanishing Cartesian components of the time-dependent quadrupole tensor, $Q_{ij}(t)$. [**10** points]

C. Express the result for $\hat{Q}(t)$ in complex form with a harmonic time-dependent part. What is the frequency of the oscillating part? What will be the frequency of the radiation it produces? What is the wavelength, λ? [**10** points]

D. Find the radiated magnetic field $\mathbf{H}$ in the radiation zone. Give the Cartesian components of $\mathbf{H}$ as functions of the angular direction θ, ϕ of the unit wave vector $\hat{n}$. [**10** points]

E. At a point on the y-axis at radius $r \gg \lambda$, what are the directions of the magnetic and electric field vectors? [**5** points]

F. Find the formula for the time-averaged power radiated per unit solid angle, $dP/d\Omega$, in the far field, as a function of the spherical angles θ, ϕ. (Hint: recall that in the radiation zone $\mathbf{E} = Z_0 \mathbf{H} \times \mathbf{n}$.) [**10** points]

A formula that may prove useful:

$$\mathbf{H} \;=\; -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\mathbf{n} \times \mathbf{Q}(\mathbf{n}) \quad (\textit{electric quadrupole radiation})$$

where the vector $\mathbf{Q}(\mathbf{n})$ is defined to have components $Q_i = Q_{ij}n_j$ with Q_{ij} the quadrupole tensor.

2 Variation on Fresnel's problem (50 points total)

The optical properties of some *topological insulators* (TI) are captured by constitutive relations which involve the fine structure constant, $\alpha = (e^2/\hbar c)/(4\pi\epsilon_0)$. With $\alpha_0 = \alpha\sqrt{\epsilon_0/\mu_0}$, the relations are

$$\mathbf{D} = \epsilon\mathbf{E} - \alpha_0\mathbf{B} \ , \ \mathbf{H} = \frac{\mathbf{B}}{\mu} + \alpha_0\mathbf{E}$$

A. Begin by writing down the Maxwell equations in matter with no free charge or current in the special case of a monochromatic plane wave. [**10** points]

B. Next insert the constitutive relations for this TI and show that a monochromatic plane wave of $(\mathbf{E}, \mathbf{B})$ is a solution of these equations. Find the wave speed of these plane waves. [**10** points]

C. A plane wave with linear polarization impinges at normal incidence on the flat surface of this TI. As a first step, write down the conditions satisfied by $\mathbf{E}$ and $\mathbf{H}$ at the interface. [**10** points]

D. Show that the transmitted wave remains linearly polarized with its electric field rotated by an angle θ_F. This is called Faraday rotation of the plane of polarization. [**20** points]

1 Throttling (30 points total)

Consider a (non-ideal) gas that has a volume V, temperature T and pressure P in the initial state. It then undergoes a Joule-Thomson process (is throttled through a porous medium). As a result of this process the pressure changes by ΔP. Assume ΔP to be small.

A. [**5** points] Is there a thermodynamic potential that is conserved during the process? If so state what it is and briefly explain why it is conserved.

B. [**10** points] Find the entropy change ΔS.

C. [**15** points] Derive the following expression for the temperature change ΔT.

$$\Delta T = \frac{1}{C_P}\left[T\left(\frac{\partial V}{\partial T}\right)_P - V\right]\Delta P.$$

2 Thermal fluctuations [20 points]

Consider a container of volume V filled with $N \gg 1$ atoms of an ideal Boltzmann gas at temperature T. Determine the probability of finding an empty void of volume $V_0 \ll V$ in the center of the container. Explain your reasoning.

3 Ideal Bose gas (50 points total)

A monoatomic ideal Bose gas is placed in a thermally isolated container of volume V. The atoms are spinless and have a mass m. The number of atoms is N. In the initial state the temperature is T, and the gas is Bose condensed, with half its atoms in the condensate.

A. [**40** points] Determine the following quantities (you may express your answers in terms of dimensionless definite integrals, which you do not need to evaluate).

- **i.** [**10** points] The condensation temperature T_0.
- **ii.** [**15** points] The energy of the gas.
- **iii.** [**15** points] The pressure.

B. [**10** points] Now consider an adiabatic expansion of the gas in which its volume changes to $2V$. Find the number of particles in the condensate in the final state. Show your work.

This exam consists of three parts, Problem 1, Problem 2 (with sections A-D) and Problem 3 (with sections A-C). Write your solutions for each section in the indicated space.

Some useful formulas:

Newton's law:

$$\vec{F} = m\vec{A}$$

Euler-Lagrange:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) = \frac{\partial L}{\partial q_a}.$$

Rotating Frame:

$$\ddot{\vec{r}} = -\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\vec{\omega} \times \dot{\vec{r}} - \vec{F}_{ext}/m$$

Moment of Inertia Tensor:

$$T_{ij} = \int d^3x\, \rho(x^2\delta_{ij} - x_i x_j).$$

Euler Equations (i, j, k cyclic, not summed over):

$$I_i\dot{\omega}_i - (I_j - I_k)\omega_j\omega_k = \tau_i.$$

Euler-angles:

$$R = R_3(\psi)R_1(\theta)R_3(\phi).$$

Hamilton's equations:

$$H(p_i, q_i) = p_i\dot{q}_i - L, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Poisson brackets:

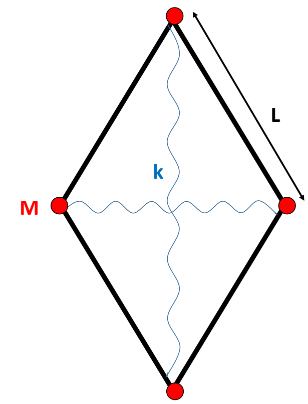
$$\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}, \qquad \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}.$$

Adiabatic Invariant / Action variable:

$$I = \frac{1}{2\pi}\oint p\,dq \qquad \text{with} \qquad p = \sqrt{2m(E - V)}.$$

1 Small oscillations (30 points)

Four massless rods of length L are hinged together at their ends to form a rhombus. A particle of mass M is attached at each joint. The opposite corners of the rhombus are joined by springs, each with a spring constant k. In the equilibrium (square) configuration the springs are unstretched. The motion is confined to a horizontal plane, and the particles only move along the diagonals of the rhombus. Introduce suitable generalized coordinates and find the Lagrangian of the system. Deduce the equations of motion and find the frequency of small oscillations about the equilibrium configuration.



2 Rigid Lamina (45 points total)

A rigid lamina is a flat rigid body with negligible thickness, that is its mass density has the form $\rho(x_1, x_2, x_3) = \delta(x_3)\mu(x_1, x_2)$. All angular velocities below are specified in the body fixed frame.

A. [**10** points] Show that for generic rigid lamina the principal moments of inertia obey $I_3 = I_1 + I_2$.

B. [**15** points] Show that for a lamina freely rotating in space the component of the angular velocity in the plane of the lamina (that is $\sqrt{\omega_1^2 + \omega_2^2}$) is constant in time.

C. [**10** points] Show that a $\vec{\omega}(t) = \omega_2 \hat{e}_2$ with constant ω_2 is a solution to the equations of motion. Study generic small perturbation around this configuration, that is consider both ω_1 and ω_3 components to be turned on with $\omega_1, \omega_3 \ll \omega_2$. Under what conditions is the solution with only ω_2 stable?

D. [**10** points] Define $\tan\alpha = \frac{\omega_2}{\omega_1}$ and derive an equation of motion for α alone.

1 Hamiltonian Dynamics and Adiabatic Invariants (25 points total)

A. [**15** points] Find the action variable I for a free particle moving in a one dimensional box of width $2a$. That is, it is subject to a potential $V(x)$=0 for $|x| \leq a$ and $V(x) = \infty$ for $|x| > a$. Derive the angular frequency $\omega = dE/dI$ of the corresponding angle variable and give a physical interpretation of your result.

B. [**5** points] Show that if the size a of the box changes adiabatically, the energy goes as $1/a^2$.

C. [**5** points] Find the time averaged force the particle applies to a single one of the walls.

Possibly useful equations, in the right context:

$$J^{\pm}|j,m\rangle = \sqrt{j(j+1) - m(m \pm 1)}\,|j,m \pm 1\rangle$$

$$r = \sqrt{x^2 + y^2 + z^2}, \quad x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

$$\left\{\frac{-\hbar^2}{2mr}\frac{d^2}{dr^2}r + \frac{\hbar^2\ell(\ell+1)}{2mr^2} + V(r)\right\}\chi(r) = E_n\chi(r)$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}, \quad Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}, \quad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{\ell -m} = (-1)^m Y^*_{\ell m}$$

$$Y_{22} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\phi}, \quad Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}, \quad Y_{20} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

$$i\hbar\dot{c_m} = \lambda\sum_n V_{mn}e^{i\omega_{mn}t}c_n, \quad \omega_{mn} \equiv \frac{E_m - E_n}{\hbar}, \quad V_{mn} \equiv \langle m|V|n\rangle$$

$$c_f^{(1)} = -\frac{i}{\hbar}\int_0^t \langle f|V|i\rangle e^{i\omega_{fi}t} \quad c_n = c_n^{(0)} + \lambda c_n^{(1)} + \cdots$$

$$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2}x + \frac{i}{(2m\hbar\omega)^{1/2}}p, \quad a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2}x - \frac{i}{(2m\hbar\omega)^{1/2}}p, \quad [a, a^\dagger] = 1$$

1 Angular momentum, Clebsch-Gordon decomposition

Consider a system consisting of the spin states of 2 particles. Particle A has total spin quantum number $s_A = 2$, and particle B has total spin quantum number $s_B = 1$. Assume the particles are in a deep trap which allows us to neglect excitation of the spatial degrees of freedom. Possible observables for this system are the spin of particle A, known as $\vec{S}_A$, the spin of particle B, known as $\vec{S}_B$, and hermitian functions of $\vec{S}_A, \vec{S}_B$. You may assume as usual that $[S_{Ai}, S_{Bj}] = 0$ for any i, j where $i, j = 1, 2, 3$ label the three components, and that $[S_{Ai}, S_{Aj}] = i\epsilon_{ijk}S_{Ak}$, $[S_{Bi}, S_{Bj}] = i\epsilon_{ijk}S_{Bk}$ (repeated indices summed). The Hamiltonian for the system is

$$H = -\gamma\vec{S}_A \cdot \vec{S}_B$$

where γ is a numerical constant.

A. [**7** points] Find the energy eigenvalues, and the degeneracy of each eigenvalue.

B. [**8** points] Find two different complete sets of compatible observables.

C. [**6** points] Define $\vec{S}_T \equiv \vec{S}_A + \vec{S}_B$. Mark each of the the following statements True or False and explain your answers.

(c1) It is possible to find a basis in which all the basis states are simultaneously eigenstates of H and eigenstates of S_{Tx} .

(c2) It is possible to find a basis in which all the basis states are simultaneously eigenstates of H and eigenstates of S_{Ax}.

D. [**9** points] Choose one of (All, Some, None) to make each of the following statements true and explain.

(d1)(All, Some, None) of the eigenstates of $\vec{S}_A \cdot \vec{S}_T$ are eigenstates of H .

(d2)(All, Some, None) of the eigenstates of H are eigenstates of $\vec{S}_{Tz}$.

(d3) (All, Some, None) of the eigenstates of $\vec{S}_{Tz}$ are eigenstates of H .

2 Wigner Eckhart Theorem and Selection rules

A. [**15** points] A certain atom is in a state $|\psi\rangle = |\alpha j m\rangle$ with total angular momentum quantum number $j = 1/2$ and $z-$component m which is either $\frac{1}{2}$ or $-\frac{1}{2}$. α is used to represent all other quantum numbers. The position operator is $\vec{r}$, the momentum operator is $\vec{p}$, and the total angular momentum operator is $\vec{J}$. The radial component of position in spherical coordinates is r, and x, y, z are the Cartesian components of $\vec{r}$.

Which of the following matrix elements can be shown to vanish using rotational symmetry arguments? Explain why or why not.

(a1)

$$\langle\alpha\ \tfrac{1}{2}\ \tfrac{1}{2}|r^2|\alpha\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

(a2)

$$\langle\alpha\ \tfrac{1}{2}\ \tfrac{1}{2}|z^2|\alpha\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

(a3)

$$\langle\alpha\ \tfrac{1}{2}\ \tfrac{1}{2}|p^2 - 3p_z^2|\alpha\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

(a4)

$$\langle\alpha\ \tfrac{1}{2}\ \tfrac{1}{2}|J_x|\alpha\ \tfrac{1}{2} - \tfrac{1}{2}\rangle$$

(a5)

$$\langle\alpha\ \tfrac{1}{2}\ \tfrac{1}{2}|J_x^2 - J_z^2|\alpha\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

B. [**5** points] Given that

$$\langle \alpha' \tfrac{1}{2}\ \tfrac{1}{2} | Y_{10} | \alpha \tfrac{1}{2}\ \tfrac{1}{2} \rangle = A$$

where A is some number you are given, use the Wigner Eckhart theorem to find the following matrix element in terms of A:

$$\langle \alpha' \tfrac{1}{2} - \tfrac{1}{2} | Y_{1-1} | \alpha \tfrac{1}{2}\ \tfrac{1}{2} \rangle = ?$$

3 Stationary Perturbation Theory [**30** points]

The Zeroth order Hamiltonian for two non identical particles of spin 1 in a magnetic field in the z direction is

$$H_0 = \mu_1 B S_{1z} + \mu_2 B S_{2z}$$

Treat the spin-spin interaction between the particles as a perturbation

$$H_1 = \gamma \vec{S}_1 \cdot \vec{S}_2$$

and find the energy levels of the system with Hamiltonian $H = H_0 + H_1$ to first order in the perturbation, assuming $|\mu_1| \neq |\mu_2|$. Would your answer change in the limit $\mu_1 = -\mu_2$? Explain why or why not.

4 Time Dependent Perturbation Theory [20 points]

A one dimensional harmonic oscillator is in its ground state at time $t = 0$. A weak field is turned on at time $t = 0$ and turned off at time $t = T$. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}\omega^2 m x^2 + V(t)$$

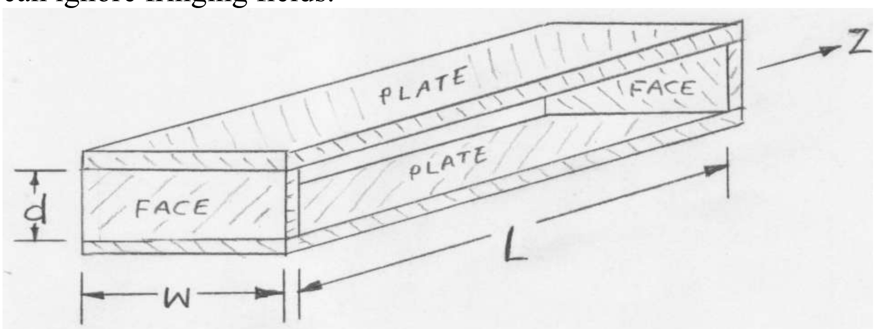
with

$$V(t) = 0, t < 0 \text{ or } t > T$$

$$V(t) = -\epsilon x^2,\ 0 < t < T\ .$$

In the weak field limit, use first order time dependent perturbation theory to find the probability that the oscillator is in the n^{th} excited state at time $t > T$, for all n for which this probability is non zero.

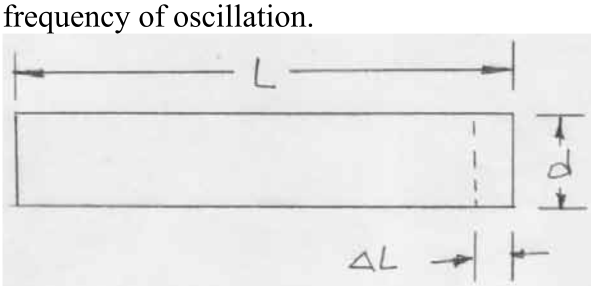
I. (35 points total) Open-sided electromagnetic resonator.
Consider an open-sided electromagnetic resonator consisting of two parallel conducting plates of width *w* separated by a gap *d*, and two parallel conducting end-faces a distance *L* apart. Consider only the lowest TEM mode of the resonator. Note $L >> w >> d$ so you can ignore fringing fields.



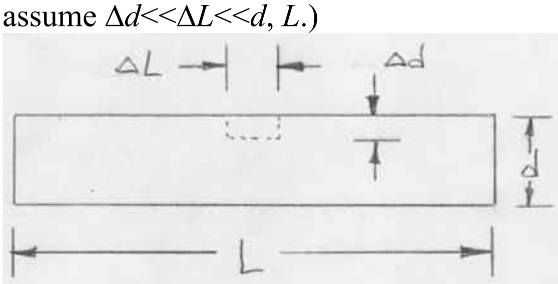
a. (5 points) Resonant frequency. What is the angular frequency of oscillation?

b. (10 points) Fields. What are the **E** and **H** fields within the resonator?

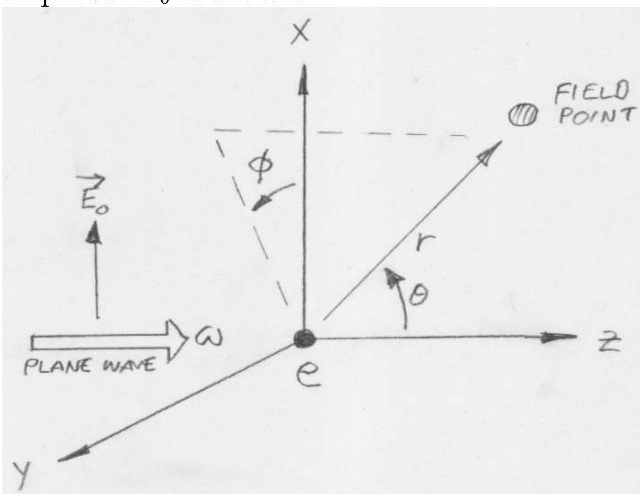
c. (10 points) Length perturbation. The resonator is perturbed in shape by moving one face slightly inwards by ΔL as shown. Find the corresponding change in the angular frequency of oscillation.



d. (10 points) Shape perturbation. The resonator is restored to length *L*. Now suppose the upper plate of the resonator is perturbed in shape by adding a conducting rectangular bar of length ΔL and height Δd across the full width *w* at the center of the resonator as shown. Find the corresponding change in the angular frequency of oscillation. (You can assume $\Delta d << \Delta L << d, L$.)



II. (30 points total) Radiation. A free point charge *e* at the origin having mass *m* is subject to a linearly-polarized plane wave of angular frequency ω with electric field amplitude $\mathbf{E}_0$ as shown.



a (10 points). Find the radiated (asymptotic) E and B fields. Assume the charge oscillates at a low enough speed where you can ignore the effects of the incident-wave's magnetic field. Recall the radiation fields emitted by an electric dipole **P** are

$\mathbf{B}(\mathbf{r},t) = -\frac{\mu_0}{4\pi}\frac{1}{rc}\hat{\boldsymbol{r}}\times\frac{\partial^2}{\partial t^2}[\mathbf{P}]_{ret}$ and $\mathbf{E}(\mathbf{r},t) = -c\hat{\boldsymbol{r}}\times\mathbf{B}(\mathbf{r},t)$

b (10 points). At a field point a distance *r* from the origin in the x-z plane and at a polar angle θ near π/4, sketch in the figure above the directions of radiation **E** and **B** fields at the field point. Describe the polarization of the radiation fields.

c (10 points) Find the intensity of the radiation fields (the time-average of the Poynting vector) in terms of the polar angle θ and the azimuthal angle ϕ.

III. (35 Points total) Reflection. A plane wave of angular frequency ω is normally incident on a conductor having permittivity and permeability that of free space and real conductivity σ. The frequency and conductivity are such that within the conductor the magnitude of conduction and displacement currents are equal.

a (10 points). Within the conductor, find the relation between conduction (true currents) and displacement currents.

b (10 points). Find the (complex) index of refraction n within the conductor.

c (15 points). Find the reflection coefficient r (the ratio of time-average incident and reflected powers).